ELSEVIER

Journal of Geometry and Physics 36 (2000) 223–250

JOURNAL OF GEOMETRY AND PHYSICS

Identification of Kähler quantizations and the Berry phase

Andrés Viña[1]

Departamento de Física, Universidad de Oviedo, Avda Calvo Sotelo, 33007 Oviedo, Spain

Received 20 July 1998; received in revised form 14 February 2000

Abstract

For a given symplectic manifold M we consider the bundle whose base is the space of Kähler structures on M, and whose fibers are the corresponding Kähler quantizations of M. We analyse the possible parallel transports in that bundle and the relation between the holonomy of some of them and the Berry phase. © 2000 Elsevier Science B.V. All rights reserved.

MSC: primary: 81S10; secondary: 53C55
Subj. Class.: Symplectic geometry
Keywords: Geometric quantization; Kähler structures

1. Introduction

To quantize a symplectic manifold (M, ω), it is necessary to choose a polarization of M. For a fixed polarization I, the corresponding quantization $\mathcal{Q}_I$ is constructed according to a well known procedure [29]. The purpose of this paper is to analyse the problem of identifying the spaces $\mathcal{Q}_I$ obtained by fixing different polarizations. This identification is one of the goals of the geometric quantization, but "the theory is far from achieving this goal" (see [7, p. 267]). We study this issue when the polarizations considered are of type Kähler, i.e., when the polarizations are complex structures on M compatible with ω. When I is a Kähler polarization, $\mathcal{Q}_I$ is the space $H^0(L_I)$ of global sections of a holomorphic line bundle L_I. This bundle is defined giving a holomorphic structure to the prequantum bundle L by means of the complex structure I. To define the referred identification, Hitchin [10] proposes to introduce a flat connection on the "quantum bundle" [1], the bundle whose fibres are the Kähler quantizations; then the parallel transport would provide an identification of the different fibres.

E-mail address: cuevas@pinon.ccu.uniovi.es (A. Viña).
[1] The author was partially supported by DGES, grant DGES-96PB0538.

PII: S0393-0440(00)00025-5

The quantization of the moduli space of flat connections on a closed surface Σ was studied in [1], and the idea proposed by Hitchin is applied to get the identification of some particular quantizations. In this case the manifold M is the symplectic quotient of an affine symplectic manifold; this fact allows to consider the complex structures on M, determined by the translationally invariant complex structures on the affine manifold. This restriction leads to a natural construction of the quantum bundle over this subspace of quantizations and to define a flat connection on it. The quantization of this moduli space was also studied in [10] in an algebraic–geometric context; here only the complex structures are considered on M which are induced by conformal structures on Σ. Over this space of special polarizations was constructed the quantum bundle and a flat connection on this bundle. Other constructions of this connection were given by Faltings [3] and Ramadas [23].

In Ref. [26] we considered the quantizations of a symplectic torus $(V/\Lambda, \omega)$ defined by the polarizations which are complex structures on the vector space V. The holomorphic line bundles on a complex torus are characterized by their Appell–Humbert data; this nice property allowed us to construct local trivializations for the bundle of quantizations. However, an identification of the quantizations is not possible if we impose that this identification should be "continuous" in a sense which will be explained below.

As it is well known, there are symplectic manifolds that do not admit Kähler structures; but if the set of Kähler structures on M is non-empty, given such a structure, the consideration of its deformations show that the space of Kähler structures is an "infinite-dimensional" object. This fact leads to conjecture the impossibility of a consistent identification of *all* Kähler quantizations of (M, ω). Here we propose a frame to study this conjecture. This proposal is in fact a development of Hitchin's idea.

In the general case, when we consider all the Kähler polarizations on an arbitrary symplectic manifold, the vector spaces $\mathcal{Q}_I$ will have different dimensions, so the desired identification of quantizations is impossible. If we restrict ourselves to the subspace $\mathcal{R}$ consisting of the complex structures for which the first Chern class of the corresponding Kähler manifold is positive, then all the vector spaces $\mathcal{Q}_I$ have the same dimension. In this case these vector spaces could be considered as fibers of a vector bundle $\cup\{\mathcal{Q}_J \,|\, J \in \mathcal{R}\}$ over $\mathcal{R}$. But among all those possible identifications of the $\mathcal{Q}_J$'s, which is the most appropriate? The polarizations are introduced in order to reduce the space $\Gamma(M, L)$ of the differentiable sections of the prequantum bundle L, since this space is too large to represent the phase space, hence, taking into account that the $\mathcal{Q}_J$'s are subspaces of $\Gamma(M, L)$, a "good" identification should be continuous in the following sense: given $\tau \in \mathcal{Q}_I$, if τ_J denotes the element of $\mathcal{Q}_J$ identified with τ, the set $\{\tau_J\}$ is a continuous family w.r.t. appropriate norms in $\Gamma(M, L)$ and in $\mathcal{R}$. Such an identification would define a transport of the fibres of $\cup\mathcal{Q}_J$ along the curves in $\mathcal{R}$, and the "curvature" of this transport would vanish. We will analyse all the possible transports of the fibres of $\cup\mathcal{Q}_J$. These transports are determined by the differential equation which they generate. That is, if I_t is a curve in $\mathcal{R}$ and $\tau_t \in \mathcal{Q}_{I_t}$ is identified with τ, one will have

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = \zeta(I_t, \tau_t),$$

where ζ is a section of L. The condition $\tau_t \in \mathcal{Q}_{I_t}$ imposes restrictions to ζ which permit us to prove the plausible result that the curvature of any such transport does not vanish. This result is, in fact, a no-go theorem for the continuous identification of the Kähler quantizations in the sense explained above.

Another point we study is the relation between our transports and the Berry phase. For a Lagrangian submanifold of M which undergoes a cyclic evolution along an isodrast is defined the corresponding Berry phase [27]. When the curve I_t in $\mathcal{R}$ is obtained from a family $\{\psi_t\}$ of Hamiltonian symplectomorphisms acting on I, the section ζ can be chosen in a natural way, and the holonomy of this transport along such a closed curve is essentially the Berry phase of a loop of antiholomorphic integral submanifolds defined by the polarizations I_t.

The space $\mathcal{K}$ of Kähler structures on (M, ω) compatible with ω is considered in Section 2. In particular, we characterize the tangent vectors to $\mathcal{K}$.

The subspace $\mathcal{R}$ of $\mathcal{K}$ is introduced in Section 3. If the canonical bundle of M_I is negative, then the Riemann–Roch theorem implies that $\dim H^0(L_I)$ is a topological invariant independent of I. Bearing this fact in mind, the space $\mathcal{R}$ is defined so that its elements determine complex structures whose canonical bundles are negative, in this way all the vector spaces $\mathcal{Q}_I$ have the same dimension, whenever $I \in \mathcal{R}$, and to rule out trivial cases we assume that this dimension is different from zero.

The transport of vectors of $\mathcal{Q}_I$ along curves in $\mathcal{R}$ is introduced in Section 4. The section ζ of L which generates a transport must satisfy a condition of cohomological nature. The existence of such sections ζ is guaranteed by the vanishing of the cohomology group $H^{0,1}(L_I)$. There are infinitely many such transports; in fact the corresponding sections ζ belong to an affine space modelled on $\mathcal{Q}_I$. A particular ζ can be constructed using the Green and the adjoint operators of $\bar{\partial}_I$, the operator which defines the holomorphic structure of L_I. This ζ determines a *canonical* transport which preserves the inner product of spaces $\mathcal{Q}_J$. In this section we also study some properties of the canonical transport along geodesics of $\mathcal{R}$.

In Section 5, we prove that canonical transport is not "flat" when $\dim M = 2n > 2$. As a consequence of this fact it can be proved that any other transport is also not flat. In the infinite-dimensional space of small closed curves in $\mathcal{R}$ we actually show the existence of curves such that the transports along them exhibits its non-vanishing curvature. Our argument does not work if $n = 1$ because the corresponding space of small closed curves is not big enough for containing, for each transport, a curve along which this transport is not flat. When $n = 1$, M_I is a Riemann surface and the elements of $\mathcal{Q}_I$ can be identified with theta functions. We show that considering the quantizations of M as spaces of theta functions, it is not possible to define a continuous identification of them. The point here is the impossibility of choosing a continuous family of characteristics [19] for the line bundles. Using this fact, we proved in [26] an analogous negative result relative to identification of the quantizations of a symplectic torus.

Section 6 is concerned with the transport along the curves generated by families of symplectomorphisms. If $\{\psi_t\}$ is a family of Hamiltonian symplectomorphisms with $\psi_0 =$ id, then $I_t = \psi_t \cdot I$ is a curve in $\mathcal{R}$. In this case there is a *natural* solution to the equation for ζ; it is defined by means of the Hamiltonian vector fields. If Q is an holomorphic integral submanifold of I, then $Q_t := \psi_t(Q)$ is an isodrastic deformation of Q (see [27]), and it turns out that the natural translation $\tau_{|Q} \to \tau_{t|Q_t}$ is flat. Thus one has a flat "translation of submanifolds" of the prequantum bundle L along isodrasts.

In Section 7, we check the natural solution for the transport obtained in Section 6 by analysing the case when the symplectic manifold is a coadjoint orbit $\mathcal{O}$ of a semisimple Lie group G. For a fixed $\eta \in \mathcal{O}$, a character Λ of the subgroup of isotropy G_η determines a prequantum bundle L. By means of a maximal torus $T \subset G_\eta$, one can define an invariant polarization I, and the elements of $\mathcal{Q}_I(L)$ can be identified with functions f on $G_{\mathbb{C}}$ which are equivariant w.r.t. Λ. If L' is the prequantum bundle determined by the character Λ' of $G_{a\eta}$ defined from Λ in a natural way, there is a *direct* identification of $\mathcal{Q}_I(L)$ with $\mathcal{Q}_I(L')$, considering these spaces as sets of equivariant functions. The torus $\tilde{T} = aTa^{-1}$ defines the polarization $\tilde{I}$ on $\mathcal{O}$, and we have the quantization $\mathcal{Q}_{\tilde{I}}(L')$. There is also a *direct* identification of $\mathcal{Q}_I(L)$ with $\mathcal{Q}_{\tilde{I}}(L')$ in terms of equivariant functions. On the other hand, given $A \in \mathfrak{g}$, let $\psi_t : \mathcal{O} \to \mathcal{O}$ be the left multiplication by e^{tA}, and $I_t = \psi_t \cdot I$ the curve generated by the family of symplectomorphisms ψ_t; we prove that the identification between $\mathcal{Q}_I$ and $\mathcal{Q}_{I_t}$ given by the natural transport is precisely the composition of the above isomorphisms $\mathcal{Q}_I(L) \simeq \mathcal{Q}_{I_t}(L_t) \simeq \mathcal{Q}_{I_t}(L)$, where L_t is the prequantum bundle determined by the character of $G_{e^{tA}\eta}$ defined by Λ.

Weinstein [27] defined the Berry phase for a Lagrangian submanifold which undergoes a cyclic evolution. In Section 8, we analyse the relation of our natural transport with the Berry phase. A particular closed curve in $\mathcal{R}$ is $I_t := \psi_t \cdot I$, where ψ_t is a family of Hamiltonian symplectomorphisms with $\psi_0 = \psi_1 = \mathrm{id}$. If P is an antiholomorphic integral submanifold of I, by means of the symplectomorphisms, one can define the loop of submanifolds $P_t = \psi_t(P)$. On the other hand, we consider the natural transport along I_t, and with τ_1 we denote the transported τ around this closed curve. Then $\tau_1(P_1) = \kappa\tau(P)$, with κ a complex number. We prove that the class of κ modulo the holonomy of P is the Berry phase of the loop of isodrastic submanifolds P_t.

In order to study the continuity of some constructions w.r.t. the complex structure I, it is necessary to have holomorphic coordinates on M_I which depend continuously on I. In Appendix A, we show the existence of such coordinates (Proposition A.2). Our proof basically consists in checking the continuity of the steps which appear in the proof of the Newlander–Nirenberg theorem. If M and I were real analytic then theNewlander–Nirenberg theorem could be deduced from Frobenius theorem [15, p. 321; 20, p. 126], and under this stronger assumption it is easy to verify the said continuity. For the general case, we have analysed the classical construction given by Kohn [17].

2. The space of Kähler polarizations

Let M be a connected, compact, symplectic C^∞ manifold of dimension $2n$, with symplectic form ω. An almost complex structure on M compatible with ω is a C^∞ section I of the vector bundle End(TM) such that

$$I^2 = -1, \tag{2.1}$$

$$\omega(I., I.) = \omega(., .), \tag{2.2}$$

$$\omega(., I.) \text{ is positive definite.} \tag{2.3}$$

If the element I has no torsion, i.e.,

$$[IX, IY] - [X, Y] - I[IX, Y] - I[X, IY] = 0$$

for all X, Y vector fields on M, then I is an integrable almost complex structure on (M, ω). By $\mathcal{K}$ we denote the set of integrable almost complex structures on M compatible with ω; we assume that $\mathcal{K} \neq \emptyset$. The elements of $\mathcal{K}$ are the Kähler polarizations of (M, ω). $\mathcal{K}$ can be endowed with a structure differentiable which in terms of inverse limits of Hilbert spaces (ILH) (see [21,22]) is a strong analytic ILH-manifold modelled on an appropriate Sobolev chain. In Ref. [4] there is a detailed exposition of this structure.

Since $\mathcal{K} \subset C^\infty(\mathrm{End}(TM))$, a tangent vector to $\mathcal{K}$ is an element of $C^\infty(\mathrm{End}(T^cM))$. If $I(t)$ is a curve in $\mathcal{K}$ with $I(0) = I$, which defines the tangent vector $C = \dot{I}(0) \in T_I\mathcal{K}$, the condition $(I(t))^2 = -1$ implies

$$IC + CI = 0. \tag{2.4}$$

From the condition (2.2) applied to $I(t)$ we deduce

$$\omega(I., C.) + \omega(C., I.) = 0. \tag{2.5}$$

If $\{z^i\}$ are local complex coordinates on M_I, we put $e_j := \partial/\partial z^j$; by (2.4)

$$C = \sum_{aj} B^a_{\bar{j}} e_a \otimes \bar{e}^j + \sum_{aj} B^{\bar{a}}_j \bar{e}_a \otimes e^j.$$

The vector

$$v_j(t) := e_j + \frac{t}{2\mathrm{i}} \sum_a B^{\bar{a}}_j \bar{e}_a$$

satisfies $(I + tC)v_j(t) = \mathrm{i}v_j(t) + \mathrm{O}(t^2)$, so $v_j(t)$ is up to order t^2 an i-eigenvector of $I(t)$. Similarly,

$$\bar{e}_j - \frac{t}{2\mathrm{i}} \sum_a B^a_{\bar{j}} e_a + \mathrm{O}(t^2)$$

is an $-$i-eigenvector of $I(t)$. Therefore $\overline{B^a_{\bar{j}}} = B^{\bar{a}}_j$, and C is determined by the element B of $\Omega^{0,1}(M, TM_I)$:

$$B = \sum_{aj} B^a_{\bar{j}} e_a \otimes \bar{\mathrm{e}}^j. \tag{2.6}$$

Taking into account (2.5), we deduce

$$\sum_a B^a_{\bar{j}} \omega_{a\bar{k}} = \sum_a B^a_{\bar{k}} \omega_{a\bar{j}},$$

hence B must have the form

$$B = \sum F^{ir} \omega_{r\bar{j}} e_i \otimes \bar{e}^j, \tag{2.7}$$

where $\sum F^{ir} e_i \otimes e_r$ is a C^∞ symmetric tensor.

The integrability condition for $I(t)$ equivalent to $[\bar{v}_j(t),\ \bar{v}_k(t)]$ is (up to order t^2) a field of type (0,1) w.r.t. $I(t)$.

$$\begin{aligned}[\bar{v}_j(t),\ \bar{v}_k(t)] &= \tfrac{1}{2}\mathrm{i}t \sum_a \left(\frac{\partial B^a_{\bar{k}}}{\partial \bar{z}^j} - \frac{\partial B^a_{\bar{j}}}{\partial \bar{z}^k} \right) e_a + \mathrm{O}(t^2) \\ &= \tfrac{1}{2}\mathrm{i}t \sum_a \left(\frac{\partial B^a_{\bar{k}}}{\partial \bar{z}^j} - \frac{\partial B^a_{\bar{j}}}{\partial \bar{z}^k} \right) v_a(t) + \mathrm{O}(t^2).\end{aligned}$$

Thus the integrability condition is equivalent to

$$0 = \bar{\partial} B \in \Omega^{0,2}(M, TM_I). \tag{2.8}$$

This condition and $\bar{\partial}\omega = 0$ implies

$$\sum \frac{\partial F^{ir}}{\partial \bar{z}^l} \omega_{r\bar{j}}\, \mathrm{d}\bar{z}^l \wedge \mathrm{d}\bar{z}^j = 0. \tag{2.9}$$

Since the tangent vectors to $\mathcal{K}$ at I are tensor fields on M_I of type (2.6), using the Kähler metric $g_I := \omega(.,I.)$, one can define an inner product on the tangent space $T_I(\mathcal{K})$ by the relation

$$(B, B') := \int_M g_I(B, B')\, \omega^n, \tag{2.10}$$

and this product defines a weak Hermitian structure on $\mathcal{K}$ [2,4].

3. Quantization

Let us suppose that ω satisfies the integrability condition [29, p. 158]. That is, ω defines a cohomology class in $H^2(M, \mathbb{R})$ which belongs to the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. Then there exists a smooth Hermitian line bundle on M whose first Chern class is $[\omega]$, and on this bundle is defined a connection compatible with the Hermitian structure and whose curvature is $-2\pi\mathrm{i}\omega$. The bundle and the connection are not uniquely determined by ω. The family of all possible pairs (line bundle, connection) can be labelled by the elements of $H^1(M, U(1))$ [29, p. 161]. From now on, unless otherwise indicated, we suppose that a "prequantum bundle" L and a connection D have been fixed, and from them we will construct the quantizations of (M, ω).

Given $I \in \mathcal{K}$, we have the corresponding decomposition in the sheaf of germs of smooth 1-forms with coefficients in L

$$\mathcal{A}^1(M_I, L) = \mathcal{A}^{1,0}(M_I, L) \oplus \mathcal{A}^{0,1}(M_I, L).$$

The operator $\frac{1}{2}(1 + \mathrm{i}I)$ extended to $\mathcal{A}^1(M_I, L)$ is a projector on $\mathcal{A}^{0,1}(M_I, L)$. We set

$$D''_I := \tfrac{1}{2}(1 + \mathrm{i}I)D : \mathcal{A}^0(M_I, L) \to \mathcal{A}^{0,1}(M_I, L).$$

A smooth section τ of L on U is said to be I-holomorphic if $D''_I\tau = 0$ on U [10].

Since $\omega(I., I.) = \omega(., .)$, ω as 2-form on M_I is of type (1,1). Consequently, if $\eta = \eta^{1,0} + \eta^{0,1}$ is the connection form of D w.r.t. a local section σ of L, then $\bar{\partial}\eta^{0,1} = 0$. By Dolbeault lemma there are local solutions to $\bar{\partial} f + f\eta^{0,1} = 0$, and thus $f\sigma$ is a local I-holomorphic section of L. These sections define an holomorphic structure on L, and the corresponding holomorphic line bundle is denoted by L_I. The space $\mathcal{Q}_I := H^0(M_I, \mathcal{O}(L_I))$ of global sections of L_I is the quantization of (M, ω) w.r.t. the "Kähler polarization" I [10].

The form ω is a positive (1,1)-form on M_I, since for $v \in TM_I$

$$-\mathrm{i}\omega(v, \bar{v}) = -\mathrm{i}\omega(Y - \mathrm{i}IY,\ Y + \mathrm{i}IY) = 2\omega(Y, IY) \geq 0.$$

Hence, L_I is a positive line bundle.

If we denote by Det_I the canonical bundle of M_I, then

$$c_1(\mathrm{Det}_I) = c_1\left(\bigwedge^n T^{*1,0}(M_I)\right) = c_1(T^{*1,0}(M_I)) = -c_1(M_I)$$

see [9, p. 64]. On the other hand, $c_1(M_I)$ can be represented by the (1,1)-form

$$\left(\frac{\mathrm{i}}{2\pi}\right)\bar{\partial}\partial \log(\det(g_{Ij\bar{k}}))$$

see [13, p. 26]. By Proposition A.2 we can conclude that if $c_1(\mathrm{Det}_I \otimes L_I^*)$ is negative, then $c_1(\mathrm{Det}_J \otimes L_J^*)$ is also negative for all J in a neighbourhood of I in $\mathcal{K}$.

If $\mathrm{Det}_I \otimes L_I^*$ is negative, $H^q(M_I, \mathcal{O}(\mathrm{Det}_I \otimes L_I^*)) = 0$ for $q < n$ by the Kodaira–Nakano vanishing theorem, and by the Serre duality [6, p. 153],

$$H^r(M_I, \mathcal{O}(L_I)) = 0 \quad \text{for } r > 0, \tag{3.1}$$

and finally the Riemann–Roch theorem implies $\dim H^0(M_I, L_I) = \chi(L_I)$, where $\chi(L_I)$ is the Euler characteristic of the line bundle L_I. Since $\chi(L_I)$ is a topological invariant, $\dim H^0(M_I, L_I)$ is independent of the complex structure. We shall assume that $\chi := \chi(L_I) > 0$.

Therefore, in order to get vector spaces $\mathcal{Q}_I$ whose dimension is independent of I, we will consider only those complex structures such that the first Chern class of the corresponding complex manifold M_I is positive; and we denote by $\mathcal{R}$ the subset of $\mathcal{K}$ consisting of these complex structures. One has the following proposition.

Proposition 1. $\dim(\mathcal{Q}_I)$ *is independent of* $I \in \mathcal{R}$.

4. The canonical transport

We are interested in "continuous" identifications of the $\mathcal{Q}_I$ which take into account that each $\mathcal{Q}_I$ is a subspace of $\Gamma(L)$ (the space of smooth sections of L), as we have explained in Section 1. The space $\cup\{\mathcal{Q}_I | I \in \mathcal{R}\}$ is a subbundle of the product bundle $\mathcal{R} \times \Gamma(M, L)$ in a *general* sense (see [11, p. 11]), but we have not defined local trivializations on it. Evidently, the application to $\cup\{\mathcal{Q}_I\}$ of some properties which hold for the standard (locally trivial) vector bundles can give rise to false conclusions. For instance, if M is a torus, then $\mathcal{R}$ is a contractible space; but it is wrong to conclude from this fact that $\cup\{\mathcal{Q}_I\}$ is trivial.

If I_t be a curve in $\mathcal{R}$ with $I_0 = I$, we want to define a "transport"

$$\tau \in \mathcal{Q}_I \subset \Gamma(L) \to \tau_t \in \mathcal{Q}_{I_t} \subset \Gamma(L),$$

which is continuous when in $\Gamma(L)$ is considered a suitable Solobev norm.

By the continuity we are assuming, the element $\tau_t \in \mathcal{Q}_{I_t}$ "identified" with τ must satisfy

$$\tau_t = \tau + t\zeta + \mathrm{O}(t^2) \in \Gamma(L), \tag{4.1}$$

where ζ is a section of the Hermitian bundle L, and $\mathrm{O}(t^2)$ is relative to an appropriate Sobolev norm $\| \|_l$. The condition $\tau_t \in \mathcal{Q}_{I_t}$ imposes restrictions to ζ. Since $\frac{1}{2}(1 + \mathrm{i}I_t)D(\tau_t) = 0$ for all t in a neighbourhood of $0 \in \mathbb{R}$, we have

$$\tfrac{1}{2}\mathrm{i}\dot{I}(0)(D\tau) + \tfrac{1}{2}(1 + \mathrm{i}I)(D\zeta) = 0.$$

Hence ζ, that depends on I, $\dot{I}(0)$ and τ, must satisfy

$$D_I''\zeta(I, \dot{I}, \tau) = -\tfrac{1}{2}\mathrm{i}\dot{I}(D_I'\tau). \tag{4.2}$$

In Proposition 2 we prove that this equation has solutions. The proof is based on two facts: the compatibility of $I(t)$ with ω, which implies (2.7); and the definition of $\mathcal{R}$, that in turn implies $H^{0,1}(L_I) = 0$ for $I \in \mathcal{R}$.

Proposition 2. *Given a curve I_t in $\mathcal{R}$ and $\tau \in \mathcal{Q}_I$, there exist solutions $\zeta(I, \dot{I}, \tau)$ to* (4.2).

Proof. In a local trivialization $D_I'\tau = \sigma \otimes \eta$, where σ is a local section of the prequantum bundle L, and η is a $(1, 0)$-form of M_I. In the following, we delete the subscript in D'' and in D'. Since $D''\tau = 0$,

$$[D'', D']\tau = D''\sigma \otimes \eta + \sigma \otimes \bar{\partial}\eta.$$

If z^i are holomorphic coordinates for M_I and $\eta = \sum \eta_i \,\mathrm{d}z^i$, then the last equation implies

$$2\pi \mathrm{i}\omega_{i\bar{l}}\tau = (\nabla_{\bar{l}}\sigma)\eta_i + \sigma \frac{\partial \eta_i}{\partial \bar{z}^l}. \tag{4.3}$$

On the other hand, the tangent vector $B := \dot{I}(0)$ has the form (2.6), and $B(D'\tau) = \sigma \otimes \beta$, with $\beta = \sum B^a_{\bar{j}}\eta_a \,\mathrm{d}\bar{z}^j$. By (2.8) one has

$$\bar{\partial}\beta = \sum B^a_{\bar{j}} \frac{\partial \eta_a}{\partial \bar{z}^l} \,\mathrm{d}\bar{z}^l \wedge \mathrm{d}\bar{z}^j.$$

Using (4.3) and (2.7), we obtain

$$D''(B(D'\tau)) = D''\sigma \wedge \beta + \sigma \otimes \bar{\partial}\beta = \tau \otimes 2\pi \mathrm{i} \sum \omega_{a\bar{l}} F^{ar} \omega_{r\bar{j}} \, \mathrm{d}\bar{z}^{l} \wedge \mathrm{d}\bar{z}^{j}.$$

Then, since F is symmetric,

$$D''(B(D'\tau)) = 0, \tag{4.4}$$

so $B(D'\tau) \in \Omega^{0,1}(M, L_I)$ is closed. On the other hand, by (3.1), $H^{0,1}(L_I) = 0$; hence $B(D'\tau)$ is exact, and there exist solutions ζ to Eq. (4.2). □

Eq. (4.2) does not have a unique solution, however the following heuristic remarks lead to a particular solution. First of all, it seems reasonable to impose the conservation of the Hermitian metrics by the transport. That is, the Hermitian structure $\langle\, ,\, \rangle$ on L induces Hermitian metrics on the L_J, then we impose $\langle \tau_t, \tau_t \rangle = \langle \tau, \tau \rangle$. Hence, follows that

$$\langle \tau, \zeta \rangle + \langle \zeta, \tau \rangle = 0.$$

Since $D''{}_I\tau = 0$, this condition holds if ζ is coexact, i.e., if

$$\zeta = \delta_I \rho, \tag{4.5}$$

where δ_I is the adjoint operator of D''_I. If we consider the Hodge decomposition for ζ, and denote by $h_I(\zeta)$ its harmonic part, then $h_I(\zeta) \in \mathcal{Q}_I$, and so the harmonic component of ζ is "redundant" in the transport of elements of $\mathcal{Q}_I$

$$\tau \in \mathcal{Q}_I \to \tau + t\zeta + \mathrm{O}(t^2) \in \mathcal{Q}_{I_t}.$$

We denote by G_I the Green operator of D''_I; if $h_I(\zeta) = 0$, by (4.5) the Hodge decomposition of ζ (see [16] or [28]) is simply $\zeta = \delta_I G_I D''_I \zeta$, and as ζ must satisfy (4.2), we get the following solution for this equation:

$$\xi(I, \dot{I}, \tau) := -\tfrac{1}{2}\mathrm{i}\delta_I G_I \dot{I}(D'{}_I\tau). \tag{4.6}$$

This particular solution to (4.2) can be obtained in a canonical way. Let $\psi := -\frac{1}{2}\mathrm{i}B(D'_I\tau)$, by (4.4) ψ is closed. As $H^1(M, L_I) = 0$, ψ is exact. Then by the Hodge decomposition of forms we can write $\psi = D''_I \delta_I G_I \psi$, therefore the section ξ given by (4.6) is a solution of (4.2). Furthermore, any other solution is

$$\zeta(I, \dot{I}, \tau) = \xi(I, \dot{I}, \tau) + h(I, \dot{I}, \tau)$$

with $h \in \Gamma(L)$ and $D''_I h = 0$. That is, the set of solutions to (4.2) is an affine space modelled on $\mathcal{Q}_I$.

In particular, $\xi(I, ., .)$ defines a transport of elements of $\mathcal{Q}_I$ along curves in $\mathcal{R}$ by the equation

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = \xi(I_t, \dot{I}_t, \tau_t). \tag{4.7}$$

Remarks.

1. *Since the Hermitian structure on L_I is given by that of the prequantum bundle, and the Kähler metric on M_I is determined by I and ω, the solution* (4.6) *to Eq.* (4.2) *has been canonically chosen. So we call the transport defined by* (4.7) *the canonical transport*.
2. *Since I_t is a smooth curve in $\mathcal{R}$ the tensor fields $\{F_t\}_t$, defined from $B_t := \dot{I}_t$ according to* (2.7)*, form a continuous family in the corresponding space. By* (4.1)*, $\{\tau_t\}_t$ is a continuous family w.r.t. appropriate Sobolev norms in $\Gamma(L)$. Hence $\{\xi(I_t, B_t, \tau_t)\}_t$ defined by* (4.6) *is a continuous family as a consequence of Proposition* A.2.
3. *If $\tau,\ \sigma \in \mathcal{Q}_I$, we have*

$$\langle \tau_t, \sigma_t\rangle = \langle \tau, \sigma\rangle + t(\langle \xi(I, B, \tau), \sigma\rangle + \langle \tau, \xi(I, B, \sigma)\rangle) + \mathrm{O}(t^2).$$

By (4.6),

$$\langle \xi(I, B, \tau), \sigma\rangle = -\tfrac{1}{2}\mathrm{i}\langle \delta G(B(D'\tau)), \sigma\rangle = -\tfrac{1}{2}\mathrm{i}\langle G(B(D'\tau)), 0\rangle = 0,$$

and similarly $\langle \tau, \xi(I, B, \sigma)\rangle = 0$. Hence this transport conserves the product of sections.

By $\mathcal{D}$ is denoted the group consisting of symplectic diffeomorphisms of (M, ω). The differential structure of this infinite-dimensional group has been studied in Ref. [4]. The action of $\mathcal{D}$ on $\mathcal{K}$ is defined as follows: given $I \in \mathcal{K}$ and $\phi \in \mathcal{D}$, then $\phi \cdot I := \phi^{-1*} I \phi^*$ (here the complex structures are considered as endomorphisms of T^*M). We will see that this action permit to identify in a natural way quantizations of (M, ω) obtained from different prequantum bundles.

Let $\phi \in \mathcal{D}$, if σ is a section of the prequantum bundle L, $\phi \cdot \sigma := \sigma \circ \phi^{-1}$ is a section of $\phi \cdot L$, the pull-back of L by ϕ^{-1}. We shall denote by $\phi \cdot D$ the connection in $\phi \cdot L$ defined by

$$((\phi \cdot D)(\phi \cdot \tau))(x) = \phi_x^{*-1}(D\tau)(\phi^{-1}(x)) \in T_x^*M \otimes L_{\phi^{-1}(x)}.$$

By $\mathcal{P}$ we denote the set of prequantum bundles (L, D) on (M, ω). The symplectomorphism ϕ determines a bijective map

$$(L, D) \in \mathcal{P} \to (\phi \cdot L, \phi \cdot D) \in \mathcal{P}.$$

The quantization defined from the prequantum bundle (L, D) by the Kähler polarization I is denoted by $\mathcal{Q}_I(L, D)$, i.e.,

$$\mathcal{Q}_I(L, D) = \{\tau \in \Gamma(L) | (1 + \mathrm{i}I)D\tau = 0\}.$$

If $\tilde{I} = \phi \cdot I$, $\tilde{L} = \phi \cdot L$, $\tilde{D} = \phi \cdot D$ for $\tau \in \mathcal{Q}_I(L, D)$ the section $\tilde{\tau} := \phi \cdot \tau$ satisfies

$$(1 + \mathrm{i}\tilde{I})\tilde{D}(\tilde{\tau})|_x = \phi_x^{*-1}(1 + \mathrm{i}I_{\phi^{-1}(x)})\phi_x^*\phi_x^{*-1}((D\tau)(\phi^{-1}(x))) = 0.$$

Hence $\tau \in \mathcal{Q}_I(L, D) \to \tilde{\tau} \in \mathcal{Q}_{\tilde{I}}(\tilde{L}, \tilde{D})$ is an isomorphism.

For $I \in \mathcal{R}$ we construct $\mathfrak{Q}_I$ the direct sum of the I-quantizations of the different prequantum bundles,

$$\mathfrak{Q}_I = \bigoplus_{\mathcal{P}} \mathcal{Q}_I(L, D),$$

and for $\tilde{I} = \phi \cdot I$, there is the corresponding induced isomorphism $\phi_I^* : \mathfrak{Q}_I \simeq \mathfrak{Q}_{\tilde{I}}$.

If we consider the exponential map defined by the inner product (2.10) and fix a normal neighbourhood U in $\mathcal{R}$ centred at I (see [2]), each point in U can be jointed to I with only one geodesic in U. By means of the canonical transport along the geodesics we can identify in a consistent way $\mathcal{Q}_J(L, D)$ with $\mathcal{Q}_I(L, D)$ for all $J \in U$. Let $I,\ \tilde{I}$ be equivalent Kähler polarizations, $\phi \cdot I = \tilde{I}$. If $\beta : V \subset M \to \mathbb{C}^n$ is a chart for M_I, which defines holomorphic coordinates (z^j), then $\tilde{\beta} = \beta \circ \phi^{-1}$ is a holomorphic chart for $M_{\tilde{I}}$ and its coordinates will be denoted by $(\tilde{z}^j)$. Moreover, $\tau \circ \beta^{-1} = \tilde{\tau} \circ \tilde{\beta}^{-1}$ for $\tau \in \mathcal{Q}_I$ and $\tilde{\tau} = \phi \cdot \tau$. Thus, if σ is a local frame for L and $\tau = f(z^j)\sigma$, then $\phi \cdot \tau = f(\tilde{z}^j)\phi \cdot \sigma$, i.e., the expression of τ in the coordinates (z^j) is the same as this one of $\tilde{\tau}$ in the coordinates $(\tilde{z}^j)$.

Proposition 3 states the invariance under symplectic diffeomorphisms of the canonical transport along geodesics.

Proposition 3. *Let I_t be a geodesic in $\mathcal{R}$ with $I_0 = I$, then $\tilde{I}_t = \phi \cdot I_t$ is also a geodesic. Moreover, if $\tau \in \mathcal{Q}_I(L, D)$ and τ_t is the transport of τ along I_t, then $\phi \cdot \tau_t$ is the transport of $\phi \cdot \tau \in \mathcal{Q}_{\tilde{I}}(\phi \cdot L, \phi \cdot D)$ along $\tilde{I}_t$.*

Proof. Since (2.10) is invariant under symplectic diffeomorphisms, if I_t is a geodesic in $\mathcal{R}$, then $\tilde{I}_t$ is geodesic too. Moreover, $\tilde{B}_t := \mathrm{d}/\mathrm{d}t(\phi \cdot I_t) = \phi^{*-1} B_t \phi^*$ with $B_t = \mathrm{d}I_t/\mathrm{d}t$; hence if $(z_t^j)_j$ are complex coordinates for M_{I_t} the functional dependence of B_t w.r.t. $(z_t^j)_j$ is the same as this one of $\tilde{B}_t$ w.r.t. $(\tilde{z}_t^j)_j$. By (4.6), $\xi(I_t, B_t, \tau_t)$ and $\xi(\tilde{I}_t, \tilde{B}_t, \phi \cdot \tau_t)$ expressed in coordinates (z_t^j) and $(\tilde{z}_t^j)$ have the same functional dependence. Therefore if τ_t is the solution to Eq. (4.7) along I_t, then $\phi \cdot \tau_t$ is the solution to the transport along $\tilde{I}_t$, i.e., $\phi \cdot \tau_t$ is the transported of $\phi \cdot \tau$ along $\tilde{I}_t$. □

If we denote by $F_I(t)$ the isomorphism between $\mathfrak{Q}_I$ and $\mathfrak{Q}_{I_t}$ determined by the canonical transport along the geodesic, then Proposition 3 asserts that $\phi_{I_t}^* F_I(t) = F_{\tilde{I}}(t)\phi_I^*$.

5. A no-go theorem

First we will prove the non-flatness of the canonical transport when $\dim M > 2$.

Proposition 4. *Assuming $\chi > 0$, if dimension of M is greater than 2, the transport defined by ξ is not flat.*

Proof. Let $\{I_t\}$ be a "small" closed curve in $\mathcal{R}$, i.e., $I_0 = I_1 =: I$ and $\|B_t := \dot{I}(t)||_l \ll 1$ for all $t \in [0, 1]$, where $\| \ \|_l$ is an appropriate Sobolev norm. We define

$$\epsilon := \sup\{\|B_t\|_l | t \in [0, 1]\}.$$

Let τ be an element of $\mathcal{Q}_I$ with $D'\tau \neq 0$. In order to study the transport defined by (4.7), we set

$$D'' = D''_I, \qquad \delta = \delta_I, \qquad G = G_I, \qquad \delta_{I_t} = \delta + \tilde{\delta}_t, \qquad G_{I_t} = G + \tilde{G}_t,$$

moreover $D'_{I_t} = D' + \frac{1}{2}\mathrm{i}(I - I_t)D$. As $\xi(I_t, \dot{I}_t, \tau_t)$ is of order ϵ, to determine the "curvature" of the transport (4.7), we first obtain the solution of the "approximate" equation

$$\frac{\mathrm{d}\sigma_t}{\mathrm{d}t} = -\tfrac{1}{2}\mathrm{i}\delta G B_t(D'\tau), \qquad \sigma_0 = \tau,$$

and then we replace on the right-hand side of (4.7) τ_t by $\sigma_t + \mathrm{O}(\epsilon^2)$.

Now we assume that I_t satisfies

$$\delta G B_t(D'\tau) = 0 \tag{5.1}$$

for all $t \in [0, 1]$. Under this hypothesis $\sigma_t = \tau$ and we obtain the following equation for τ_t:

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = -\tfrac{1}{2}\mathrm{i}(\delta + \tilde{\delta}_t)(G + \tilde{G}_t)B_t(D' + \tfrac{1}{2}\mathrm{i}(I - I_t)D)\tau + \mathrm{O}(\epsilon^3).$$

By (5.1) and $D''\tau = 0$, we conclude $\delta G B_t I(D\tau) = \mathrm{i}\delta G B_t(D'\tau) = 0$. So

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = -\tfrac{1}{2}\mathrm{i}\tilde{\delta}_t G B_t(D'\tau) - \tfrac{1}{2}\mathrm{i}\delta\tilde{G}_t B_t(D'\tau) - \tfrac{1}{4}\delta G B_t I_t(D'\tau) + \mathrm{O}(\epsilon^3).$$

We need Lemma 5 whose proof will be given at the end.

Lemma 5. *The term $\delta G B_t I_t(D'\tau)$ is of order ϵ^3.*

By Lemma 5,

$$\tau(1) - \tau = -\frac{\mathrm{i}}{2}\int_0^1 \tilde{\delta}_t G B_t(D'\tau)\,\mathrm{d}t - \frac{\mathrm{i}}{2}\int_0^1 \delta\tilde{G}_t B_t(D'\tau)\,\mathrm{d}t + \mathrm{O}(\epsilon^3).$$

We put $\alpha_t := B_t(D'\tau)$, so $\delta_t G_t \alpha_t = (\tilde{\delta}_t G + \delta\tilde{G}_t)\alpha_t + \mathrm{O}(\epsilon^3)$.

$$\tau(1) - \tau = -\frac{\mathrm{i}}{2}\int_0^1 (\delta_t G_t \alpha_t)\,\mathrm{d}t + \mathrm{O}(\epsilon^3) = -\tfrac{1}{2}\mathrm{i}\delta_u G_u \alpha_u + \mathrm{O}(\epsilon^3)$$

with $u \in [0, 1]$. Therefore, in general, $\tau(1) \neq \tau$. □

Proof of Lemma 5.

$$D'\tau = \rho_t^{1,0} + \rho_t^{0,1} \in \Omega^{1,0}(M, L_{I_t}) \oplus \Omega^{0,1}(M, L_{I_t}),$$

where $\rho_t^{0,1} = -\frac{1}{2}\mathrm{i}(-I + I_t)(D'\tau)$, and so $\rho_t^{0,1}$ is of order ϵ. Similarly,

$$\alpha_t = \alpha_t^{0,1} + \alpha_t^{1,0},$$

and since $B(D'\tau) \in \Omega^{0,1}(M, L_I)$ and $\mathrm{O}(\|B_t\|) = \epsilon$, then

$$\mathrm{O}(\|\alpha_t^{0,1}\|) = \epsilon, \qquad \mathrm{O}(\|\alpha_t^{1,0}\|) = \epsilon^2.$$

Next we decompose

$$\alpha_t^{1,0} = \beta_t^{1,0} + \beta_t^{0,1} \in \Omega^{1,0}(M, L_I) \oplus \Omega^{0,1}(M, L_I).$$

The orders of $\beta_t^{1,0}$ and $\beta_t^{0,1}$ are ϵ^2 and ϵ^3, respectively. We have an analogous decomposition $\alpha_t^{0,1} = \gamma_t^{1,0} + \gamma_t^{0,1}$.

On the other hand, by (5.1) $\delta G(\alpha_t)$ vanishes, hence so do its component in $\Omega^{0,1}(M, L_I)$, i.e.,

$$\delta G\beta_t^{0,1} + \delta G\gamma_t^{0,1} = 0. \tag{5.2}$$

Taking into account that $I_t B_t = -B_t I_t$ and (5.2),

$$\delta G B_t I_t(D'\tau) = \mathrm{i}\delta G(\alpha_t^{0,1} - \alpha_t^{1,0}) = \mathrm{i}G\delta(\gamma_t^{0,1} - \beta_t^{0,1}) = -2\mathrm{i}G\delta(\beta_t^{0,1}).$$

Thus, $\delta G B_t I_t(D'\tau)$ is of order ϵ^3. □

Remark. *Differentiating condition* (5.1) *at* $t = 0$, *we obtain that* $\delta B(D'\tau) \in \Omega^0(M, L_I)$ *is harmonic. From* $D''\delta B(D'\tau) = 0$, *one deduces that* $\delta B(D'\tau) = 0$. *Then by* (4.4), *the* (0, 1)*-form* $B(D'\tau) \in \Omega^{0,1}(M, L_I)$ *is harmonic, and by* (3.1) *it vanishes. With the notations of Proposition* 2, *the condition* $B(D'\tau) = 0$ *is expressed in local coordinates*:

$$\sigma \otimes \sum F^{ir}\omega_{r\bar{j}}\eta_i \,\mathrm{d}\bar{z}^j = 0.$$

Therefore, assumption (5.1) on I_t implies this linear condition on the $\frac{1}{2}n(n+1)$ functions F^{ir}. For $n = 1$ there is only one function, so in general $B(D'\tau) = 0$ implies $B = 0$.

If $n > 1$, condition (5.1) is satisfied by an "infinite-dimensional" family $\mathcal{F}$ of "small" closed curves in $\mathcal{R}$, and for a generic curve in $\mathcal{F}$, the transport defined by (4.7) is not flat. Note that in this case there is no finite-dimensional subspace of $T_I(\mathcal{R})$ containing the set $\{\dot{I}(0) | \{I_t\} \in \mathcal{F}\}$.

We next analyse the transport defined by a general solution $\zeta = \xi + h$ of (4.2) when $n > 1$. We recall that $h(I, \dot{I}, \tau) \in \mathcal{Q}_I$. Let us suppose that we can choose a continuous family

$$\{h(I, B, \rho) | B \in T_I(\mathcal{R}), \quad \rho \in \mathcal{Q}_I\}.$$

As the right-hand side of (4.2) and ξ are linear in B and τ, so is h. When B runs over $T_I(\mathcal{R})$ and ρ runs over $\mathcal{Q}_I$, then $h(I, B, \rho)$ varies in the finite-dimensional vector space $\mathcal{Q}_I$. For each $\tau \in \mathcal{Q}_I$, with $D'\tau \neq 0$ we have a linear map $h(I, ., \tau) : T_I(\mathcal{R}) \to \mathcal{Q}_I$, and codim $\mathrm{Ker}(h(I, ., \tau)) \leq \dim \mathcal{Q}_I = \chi$. Hence, there exists a subspace $\mathcal{B}$ of $T_I(\mathcal{R})$ with codim $\mathcal{B} < \infty$, such that $h(I, B, \tau) = 0$ for all $B \in \mathcal{B}$. We conclude that for a generic curve in $\mathcal{R}$ the transports of τ defined by ξ and by $\zeta = \xi + h$ along this curve are equal. By Proposition 4, we can state the following no-go theorem for a continuous identification of Kähler quantizations of (M, ω).

Theorem 6. *If* $\chi > 0$ *and* $\dim M > 2$, *the transport defined by any solution of Eq.* (4.2) *is not flat*.

Next we study the case when dim $M = 2$. Now M_I is a Riemann surface, and to exclude the trivial case we suppose that its genus g is greater than 0. The elements of $\mathcal{Q}_I = H^0(M, L_I)$ can be identified with "theta functions" on the universal covering space $\tilde{M}$ of M. Therefore, it is reasonable to impose to the identification the continuity of the family $\{\tau_t\}_t$ in the space of holomorphic functions on $\tilde{M}$.

More precisely, given a closed curve I_t in $\mathcal{R}$ and $\tau \in \mathcal{Q}_I$, we pose the following question: is it possible to define a family $\{\tau_t \in \mathcal{Q}_{I_t}\}$ with $\tau_0 = \tau$, and for each t a theta function $\tilde{\tau}_t$, associated with τ_t, so that $\{\tilde{\tau}_t\}_t$ is a continuous family of analytic functions on $\tilde{M}$ w.r.t. an appropriate Sobolev norm?

We first summarize the well-known construction of the space of theta functions (for a detailed exposition, see [8]). Fixed a marking on M at the point p, one has a basis $A_1, \dots, A_g, B_1, \dots, B_g$ for $H_1(M, \mathbb{Z})$. For each $c \in \mathbb{C}^g$, one defines χ_c a character of the covering translation group Γ of $\tilde{M}$ by the relations

$$\chi_c(A_j) = 1, \qquad \chi_c(B_j) = \mathrm{e}^{2\pi \mathrm{i} c_j}.$$

By λ, we denote a factor of automorphy of the divisor $r \cdot p$, where r is $\int_M \omega$. Given a holomorphic line bundle L' over M_I with Chern class r, L' has a factor of automorphy of the form $\chi_c \lambda$; furthermore c is unique up to translation by elements of the lattice $\Lambda = (Id, \Omega)\mathbb{Z}^{2g}$, where Ω is the standard period matrix associated to the fixed basis for $H_1(M, \mathbb{Z})$ (see [8, Chapter II]). We call c a characteristic for L', in accordance with the denomination used in [19] in the case $g = 1$. The space $H^0(M, L')$ can be identified with the space Th_c of holomorphic functions f on $\tilde{M}$ such that

$$f(T \cdot z) = \chi_c(T)\lambda(T, z) f(z)$$

for all $z \in \tilde{M}$ and $T \in \Gamma$.

If $I \in \mathcal{R}$, by Riemann–Roch theorem dim $H^0(M, L') = r + 1 - g$. Then there exist $r + 1 - g$ holomorphic functions f_j on $\mathbb{C}^g \times \tilde{M}$ such that $\{f_j(c, .)\}_j$ is a basis for Th_c (see Theorem 7 in Section 7 of [8]). This basis gives rise in turn to a vector valued function

$$\Theta : \mathbb{C}^g \times \tilde{M} \to \mathbb{C}^{r+1-g},$$

that in [8] is called a generalized theta function of rank $r + 1 - g$.

If $\{I_t\}_{t \in S^1}$ is a closed curve in $\mathcal{R}$, a consistent and continuous identification of the $\mathcal{Q}_{I_t}$ as spaces of theta functions implies the construction of a continuous family $\{c_t\}_{t \in S^1}$ for the respective L_{I_t}. We shall show that this is, in general, impossible.

The definition of a characteristic c_t for L_{I_t} involves the following steps:

1. Let L_t^0 be the bundle on M_{I_t} defined by the divisor $r \cdot p$; then the first Chern class of $L_{I_t} \otimes (L_t^0)^{-1}$ is zero.
2. By the weak form of Abel's theorem (see [8]), $L_{I_t} \otimes (L_t^0)^{-1}$ has a factor of automorphy ρ_t which is constant on $\tilde{M}$, i.e., $\rho_t \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$. It is straightforward to check that the family $\{\rho_t\}_t$ depends continuously on the parameter t. A proof of this fact when $g = 1$ is given in Ref. [26].
3. One can write $\rho_t(T) = \mathrm{e}^{2\pi \mathrm{i}\sigma_t(T)}$, with $\sigma_t \in \mathrm{Hom}(\Gamma, \mathbb{C})$, then a characteristic c_t for L_{I_t}

is given by

$$c_{tj} = \sigma_t(B_j) - \sum_1^g \sigma_t(A_i)\Omega_{ij},$$

i.e., c_t is determined by σ_t.

Thus, we have a continuous map $\rho : S^1 \times \Gamma \to \mathbb{C}^*$ with $\rho(t, T) = \rho_t(T)$. But $S^1 \times \Gamma$ is not simply connected, so in general it is not possible to define a continuous map $\sigma : S^1 \times \Gamma \to \mathbb{C}$ such that $\rho = \mathrm{e}^{2\pi \mathrm{i}\sigma}$. Therefore, we have the following proposition.

Proposition 7. *If* $\dim M = 2$ *and its genus is greater than* 0, *there are closed curves* $\{I_t\}$ *in* $\mathcal{R}$ *for which it is impossible to define a continuous family* $\{c_t\}_t$ *of characteristics for the* L_{I_t}.

Under the hypotheses of Proposition 7, considering the Kähler quantizations of M as spaces of theta functions, it is impossible to define a consistent continuous identification of them.

6. Transport along isodrasts

Let $\{\psi_t\}$ be a family of Hamiltonian symplectomorphisms of (M, ω) generated by the time-dependent Hamiltonian f_t. We assume that $\psi_0 = \mathrm{id}$, and denote by X_t the vector fields defined by

$$\iota_{X_t}\omega = -\mathrm{d}f_t. \tag{6.1}$$

Given the Kähler structure $I \in \mathcal{R}$, one defines $I_t = \psi_t \cdot I = (\psi_t^{-1})^* I \psi_t^*$. The tangent vector $\dot{I}(0)$ to I_t at $t = 0$ can be easily obtained recalling that $\mathrm{d}\psi_t/\mathrm{d}t = X_t \circ \psi_t$. In fact,

$$\dot{I}(0) = I\left(\frac{\mathrm{d}\psi_t^*}{\mathrm{d}t}\right)_{t=0} - \left(\frac{\mathrm{d}\psi_t^*}{\mathrm{d}t}\right)_{t=0} I = I\,\mathrm{d}X_0 - \mathrm{d}X_0 I,$$

where $\mathrm{d}X_0$ is the 1-form *TM*-value obtained from X_0 by exterior differentiation. If z^a are holomorphic coordinates on M_I, and $X_0 = \sum(X^a(\partial/\partial z^a) + X^{\bar{a}}(\partial/\partial \bar{z}^a))$, it is immediate to check that

$$\dot{I}(0)(\mathrm{d}z^a) = -2\mathrm{i}\sum \frac{\partial X^a}{\partial \bar{z}^c}\,\mathrm{d}\bar{z}^c.$$

Similarly,

$$\dot{I}(0)(\mathrm{d}\bar{z}^a) = 2\mathrm{i}\sum \frac{\partial X^{\bar{a}}}{\partial z^c}\,\mathrm{d}z^c,$$

hence $\dot{I}(0) = -2\mathrm{i}\bar{\partial}(X_0^{1,0}) + 2\mathrm{i}\bar{\partial}(X_0^{0,1})$. In general,

$$\dot{I}(t) = -2\mathrm{i}\bar{\partial}_t(X_t^{1,0}) + 2\mathrm{i}\bar{\partial}_t(X_t^{0,1}),$$

where $\bar{\partial}_t$ is the Dolbeault operator determined by the complex structure I_t, and $X_t^{1,0},\ X_t^{0,1}$ are the components of X_t in the direct sum $T^{1,0}(M_{I_t}) \oplus T^{0,1}(M_{I_t})$.

Therefore,

$$-\tfrac{1}{2}\mathrm{i}\dot{I}(t)(D'_t\tau_t) = -\bar{\partial}_t(X_t^{1,0})(D'_t\tau_t). \tag{6.2}$$

In a local trivilization $D'_t\tau_t = \sigma \otimes \eta$ with η (1,0)-form w.r.t. I_t and σ section of L. A direct calculation gives

$$(\bar{\partial}_t(X_t^{1,0}))(D'_t\tau_t) = \sigma \otimes \bar{\partial}_t(\eta(X_t)) - \sigma \otimes \sum \frac{\partial \eta_a}{\partial \bar{z}_t^j} X_t^a \,\mathrm{d}\bar{z}_t^j, \tag{6.3}$$

z_t^a being holomorphic coordinates on M_{I_t}, X_t^a are the components of X_t w.r.t. these coordinates, and $\eta = \sum \eta_a \,\mathrm{d}z_t^a$. As in the proof of Proposition 2,

$$[D''_t, D'_t]\tau_t = D''_t(\sigma \otimes \eta) = D''_t\sigma \otimes \eta + \sigma \otimes \bar{\partial}_t\eta.$$

If $\omega = \sum \omega_{ta\bar{j}} \,\mathrm{d}z_t^a \wedge \mathrm{d}\bar{z}_t^j$, we have

$$2\pi\mathrm{i}\omega_{ta\bar{j}}\tau_t = (\nabla_{t\bar{j}}\sigma)\eta_a + \sigma\frac{\partial \eta_a}{\partial \bar{z}_t^j}.$$

Hence,

$$\sigma \otimes \sum X_t^a \frac{\partial \eta_a}{\partial \bar{z}^j}\mathrm{d}\bar{z}^j = 2\pi\mathrm{i}\tau_t \otimes i_{X_t^{1,0}}\omega - \eta(X_t)D''\sigma,$$

so (6.2) can be written as

$$-\tfrac{1}{2}\mathrm{i}\dot{I}_t(D'_t\tau_t) = -\sigma \otimes \bar{\partial}_t(\eta(X_t)) + 2\pi\tau_t \otimes i_{X_t^{1,0}}\omega - \eta(X_t)(D''\sigma). \tag{6.4}$$

On the other hand,

$$\zeta_t := -2\pi\mathrm{i} f_t\tau_t - D_{X_t}\tau_t \tag{6.5}$$

satisfies

$$D''_t\zeta_t = 2\pi\mathrm{i}\tau_t \otimes i_{X_t^{1,0}}\omega - \eta(X_t)D''_t\sigma - \sigma \otimes \bar{\partial}_t(\eta(X_t)).$$

This expression coincides with (6.4). Thus, we have proved the following proposition.

Proposition 8. $\zeta_t := -2\pi\mathrm{i} f_t\tau_t - D_{X_t}\tau_t$ *is solution to* (4.2) *for the transport along the curve,* $I_t = (\psi_t^{-1})^* I \psi_t^*$, *where* $\mathrm{d}\psi_t/\mathrm{d}t = X_t \circ \psi_t$, *and* $i_{X_t}\omega = -\mathrm{d}f_t$.

The solution to (4.2) defined in (6.5) will be called the *natural* solution, and the corresponding transport the natural transport along I_t.

Remark. *The family* $\{\psi_t\}$ *determines uniquely the vector fields* X_t, *but the function* f_t *is determined by* X_t *up to an additive real constant. If we consider* $f'_t = f_t + c_t$, *the corresponding* ζ'_t *differs from* ζ_t *in a multiple of* τ_t. *Hence, the transports between the projective spaces* $\mathbb{P}\mathcal{Q}_I \to \mathbb{P}\mathcal{Q}_{I_t}$ *induced by* ζ_t *and* ζ'_t *are equal.*

Let us consider two families $\{\psi_t\}$ and $\{\phi_\epsilon\}$ of Hamiltonian symplectomorphisms with $\psi_0 = \phi_0 = \mathrm{id}$, generated by the time-dependent Hamiltonians f_t and g_ϵ, respectively.

We put $f := f_0,\ g := g_0,\ X := X_0$ and $Y := Y_0$, where X_t and Y_t are the vector fields generated by f_t and g_t, respectively. Let $I_t = \psi_t \cdot I$, and with τ_t we denote the transported of $\tau \in \mathcal{Q}$ along I_t, so $\tau_t = \tau + t(-2\pi \mathrm{i} f\tau - D_X\tau) + \mathrm{O}(t^2)$. For a fixed value $|t| \ll 1$ we put $I' := I_t$ and $\tau' = \tau_t$; now we consider the curve $J'_\epsilon = \phi_\epsilon \cdot I'$, the transported of τ' along J'_ϵ is $\tau' - \epsilon(2\pi \mathrm{i} g\tau' + D_Y\tau') + \mathrm{O}(\epsilon^2)$, whose term of order $t\epsilon$ is

$$4\pi^2\mathrm{i}^2 gf\tau + 2\pi\mathrm{i}(gD_X\tau + Y(f)\tau + fD_Y\tau) + D_Y D_X\tau. \tag{6.6}$$

If $\phi_\epsilon \cdot (\psi_t \cdot I) = \psi_t \cdot (\phi_\epsilon \cdot I)$ we can construct a small closed curve in the space $\mathcal{R}$ and the curvature of this transport is the difference between (6.6) and the expression obtained from (6.6) by exchanging X with Y and f with g. That is,

$$2\pi\mathrm{i}(Y(f) - X(g))\tau + (D_Y D_X - D_X D_Y)\tau = 2\pi\mathrm{i}\omega(Y, X)\tau + D_{[Y,X]}\tau. \tag{6.7}$$

Therefore, the natural transport is not flat.

We will consider this non-flatness from another point of view. One can associate to each C^∞ function f on M a linear operator O_f on the space $\Gamma(L)$ defined by $\mathrm{O}_f(\sigma) = -2\pi\mathrm{i} f\sigma - D_{X_f}\sigma$. It is straightforward to check $\mathrm{O}_{\{f,g\}} = \mathrm{O}_f \circ \mathrm{O}_g - \mathrm{O}_g \circ \mathrm{O}_f$; the Poisson bracket $\{f, g\}$ is defined as $\omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields associated to f and g, respectively. So one has a representation of the Lie algebra $C^\infty(M)$. On the other hand, in the algebra of linear operators on $\Gamma(L)$, one can consider the ideal $\mathbb{C}$ consisting of the operators multiplication by a constant, this allows us to define a representation of the algebra $\Xi_H(M)$ of the Hamiltonian vector fields on M in the algebra $\mathrm{End}(\Gamma(L))/\mathbb{C}$.

$$\Xi_H(M) \to \mathrm{End}(\Gamma(L))/\mathbb{C}. \tag{6.8}$$

$\Xi_H(M)$ is the Lie algebra of the group $\mathrm{Ham}(M)$ of Hamiltonian symplectomorphisms. There are obstructions of topological nature to extend representation (6.8) to a projective representation

$$\mathrm{Ham}(M) \to PL(\Gamma(L))$$

of the group $\mathrm{Ham}(M)$. The fact that (6.7) does not vanish is a manifestation of these obstructions.

However, if we restrict the sections τ_t to the holomorphic submanifolds of M_{I_t}, the natural transport is flat. In fact, if Q is a holomorphic integral submanifold of the Kähler polarization I, then $Q_t = \psi_t(Q)$ is a holomorphic integral submanifold of I_t. If the vector field X_t is of type $(0, 1)$ on Q_t w.r.t. I_t, then the family $\{Q_t\}$ of Lagrangian submanifolds is an isodrast; an isodrastic deformation of Q [27]. Let τ_t be the transported of τ by means of (6.5). One can consider the transport of the restrictions $\tau_{|Q} \to \tau_{t|Q_t}$. If X, Y are of type $(0, 1)$ on Q, then for all $q \in Q$, $\omega(X, Y)(q) = 0$, since ω is of type $(1, 1)$. Moreover, $[X, Y](q)$ is also of type $(0, 1)$ by the integrability of I, so $(D_{[X,Y]}\tau)(q) = 0$, since $D''\tau = 0$. Hence (6.7) vanishes at q. In summary, the following proposition is stated.

Proposition 9. *The natural transport $\tau_{|Q} \to \tau_{t|Q_t}$ along an isodrast has curvature zero.*

7. Example: coadjoint orbits

In this section we analyse the natural transport when the symplectic manifold is a coadjoint orbit of a semisimple Lie group G. On a fixed orbit one can consider *invariant* prequantizations and polarizations. The elements of the corresponding quantizations can be viewed as spaces of equivariant holomorphic functions on $G_{\mathbb{C}}$. This fact allows a *direct* identification between these particular quantizations of the orbit. We shall check that the identification given by the natural transport and the direct one are equal.

We recall some properties about the coadjoint action of a Lie group G on $\mathfrak{g}^*$, the dual of its Lie algebra (for details see [12,18]). Given $g \in G$ and $A \in \mathfrak{g}$ with gA, we denote $\mathrm{Ad}_g(A)$; and for $\eta \in \mathfrak{g}^*$, $(g\eta)(A) := \eta(g^{-1}A)$. The invariant vector field on $\mathfrak{g}^*$ defined by A is denoted with X_A, i.e., $X_A(\eta)$ is defined by the curve $\mathrm{e}^{tA}\eta$. Through the isomorphism $T_\eta\mathfrak{g}^* \simeq \mathfrak{g}^*$, one has

$$X_A(\eta)(C) = \eta([C, A]) \quad \text{for every } C \in \mathfrak{g}. \tag{7.1}$$

Given $a \in G$, we denote by l_a the diffeomorphism of $\mathfrak{g}^*$ defined by the product of a. As a consequence of the invariance of X_A, one has the following proposition.

Proposition 10. $(l_a)_*(X_A(\eta)) = X_{aA}(a\eta)$.

Proof. By the definitions and (7.1),

$$\begin{aligned}(l_a)_*(X_A\eta)(C) &= \frac{\mathrm{d}}{\mathrm{d}t}(\eta((a\mathrm{e}^{tA})^{-1}C))|_{t=0} \\ &= -\eta([A, a^{-1}C]) = (a\eta)[C, aA] = X_{aA}(a\eta)(C). \qquad \square\end{aligned}$$

We will denote by $\mathcal{O}$ the orbit of $\eta \in \mathfrak{g}^*$ under the coadjoint action. On $\mathcal{O}$ one defines the 2-form ω

$$\omega_\mu(X_A(\mu), X_C(\mu)) = \mu([A, C]).$$

This form satisfies $l_a^*\omega = \omega$. Moreover for $A \in \mathfrak{g}$, one defines $h_A \in C^\infty(\mathcal{O})$ by the relation $h_A(\mu) = \mu(A)$; and for this function holds the formula

$$i_{X_A}\omega = \mathrm{d}h_A. \tag{7.2}$$

It is easy to prove that ω defines a symplectic structure on $\mathcal{O}$.

The orbit $\mathcal{O}$ can be identified with the quotient G/G_η, where G_η is the subgroup of isotropy of η, whose Lie algebra is

$$\mathfrak{g}_\eta = \{A \in \mathfrak{g} | \eta([A, B]) = 0 \quad \text{for all } B \in \mathfrak{g}\}. \tag{7.3}$$

On the other hand, if $\eta, \eta' \in \mathfrak{g}^*$ and $\eta' = a\eta$ with $a \in G$, then

$$G_{\eta'} = aG_\eta a^{-1}. \tag{7.4}$$

Invariant polarizations. If G is a compact semisimple Lie group and T a maximal torus, we denote by $\mathfrak{h}$ the Lie algebra of T. By the standard theory about the structure of semisimple Lie algebras (see [5,24]),

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \mathfrak{R}} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{R}$ is the corresponding set of roots, each $\mathfrak{g}_\alpha$ has dimension 1 and $\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{-\alpha}$. If $\mathfrak{R}^+$ is a system of positive roots, and E_α is a basis for $\mathfrak{g}_\alpha$, then

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathfrak{R}^+} (\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha) \right)$$

with $A_\alpha = \mathrm{i}(E_\alpha + E_{-\alpha})$ and $B_\alpha = E_\alpha - E_{-\alpha}$. We take $Z_\alpha = [B_\alpha, A_\alpha]$; the vector $Z_\alpha \in \mathfrak{h}$ is characterized by $\alpha(Z_\alpha) = 2\mathrm{i}$.

The definition of invariant polarizations for $\mathcal{O}$ follows a well-known process. Here we recall those aspects that we shall need below (for details, see [29]). Let T be a maximal torus in G such that $T \subset G_\eta$, then for $Y \in \mathfrak{g}_\alpha$, by (7.3),

$$0 = \eta([Z_\alpha, Y]) = \alpha(Z_\alpha)\eta(Y) = 2\mathrm{i}\eta(Y),$$

hence η extended to $\mathfrak{g}_{\mathbb{C}}$ vanishes on $\oplus \mathfrak{g}_\alpha$. We define $\mathfrak{R}^+_\eta = \{\alpha \in \mathfrak{R} | \eta(Z_\alpha) > 0\}$, and let us assume that $\mathfrak{R}^+$ has been chosen so that $\mathfrak{R}^+_\eta \subset \mathfrak{R}^+$. It is easy to prove that

$$\mathfrak{g}_\eta = \mathfrak{h} \oplus \left(\bigoplus_{\alpha} (\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha) \right),$$

where α runs over $\mathfrak{R}^+ - \mathfrak{R}^+_\eta$. One defines

$$\mathfrak{p} = \mathfrak{h}_{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \notin \mathfrak{R}^+_\eta} \mathfrak{g}_\alpha \right), \tag{7.5}$$

$\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which corresponds to a parabolic subgroup P of $G_{\mathbb{C}}$. The tangent space $T_\eta \mathcal{O}$ of the orbit $\mathcal{O}$ at the point η is isomorphic to $\mathfrak{g}/\mathfrak{g}_\eta$, and from the above relations, one deduces that

$$\mathfrak{g}/\mathfrak{g}_\eta = \bigoplus_{\alpha \in \mathfrak{R}^+_\eta} (\mathbb{R}A_\alpha \oplus \mathbb{R}B_\alpha).$$

Then

$$T^{\mathbb{C}}_\eta \mathcal{O} = \bigoplus_{\alpha \in \mathfrak{R}^+_\eta} (\mathfrak{g}_\alpha \oplus \bar{\mathfrak{g}}_\alpha),$$

and there is an invariant almost complex structure on $\mathcal{O}$ in which the vectors $X_C(\mu)$ with C in $\oplus_{\alpha \in \mathfrak{R}^+_\eta} \mathfrak{g}_\alpha$ span the space $T^{1,0}_\mu \mathcal{O}$. Since

$$\mathfrak{n} := \bigoplus_{\alpha \in \mathfrak{R}^+_\eta} \mathfrak{g}_\alpha$$

is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$, this almost complex structure is in fact integrable. The corresponding endomorphism I of $T_\eta\mathcal{O}$ is given by $I(A_\alpha) = -B_\alpha$, $I(B_\alpha) = A_\alpha$. This complex structure is also compatible with the symplectic form ω.

If we choose another maximal torus $\tilde{T}$, it is conjugate with T, $\tilde{T} = aTa^{-1}$. By (7.4), $\tilde{T}$ is contained in $G_{a\eta}$. We denote by $\tilde{\mathfrak{R}}$ the set of roots relative to $\tilde{T}$. Ad_a is an isomorphism between $\mathfrak{h}$ and $\tilde{\mathfrak{h}} := \mathrm{Lie}(\tilde{T})$, and if $\alpha \in \mathfrak{R}$, then $\tilde{\alpha} := \alpha \circ \mathrm{Ad}_{a^{-1}}$ is an element of $\tilde{\mathfrak{R}}$; moreover the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\tilde{\alpha}}$ are related by Ad_a, i.e.,

$$\mathfrak{g}_{\tilde{\alpha}} = \mathrm{Ad}_a(\mathfrak{g}_\alpha). \tag{7.6}$$

The torus $\tilde{T}$ allows us to define a new complex structure on $\mathcal{O}$ in which the corresponding space of $(1, 0)$ vectors at $\mu \in \mathcal{O}$, say $\tilde{T}^{1,0}_\mu\mathcal{O}$, is the space spanned by $X_C(\mu)$, with $C \in \oplus_{\tilde{\alpha}\in\mathfrak{R}^+_{a\eta}} \mathfrak{g}_{\tilde{\alpha}}$. By Proposition 10 and (7.6), $(l_a)_*$ applies $T^{1,0}_\eta\mathcal{O}$ in $\tilde{T}^{1,0}_{a\eta}\mathcal{O}$. Hence, we have the following proposition.

Proposition 11. *The invariant complex structure on $\mathcal{O}$ determined by the maximal torus aTa^{-1} is $\tilde{I} := l_{a*} \circ I \circ l^{-1}_{a*}$, where I is the complex structure defined by T.*

Quantization of $\mathcal{O}$. The orbit $\mathcal{O}$ of η possesses a G-invariant prequantization iff the linear functional

$$\rho_\eta : C \in \mathfrak{g}_\eta \to 2\pi \mathrm{i}\eta(C) \in \sqrt{-1}\mathbb{R}$$

is integral, in the sense that there is a character $\Lambda_\eta : G_\eta \to U(1)$ whose derivative is ρ_η (see [18]). Henceforth, we assume that ρ_η is integral. The corresponding prequantum bundle $L \equiv L_\lambda$, considered as a bundle on G/G_η, is precisely

$$L = G \times_\Lambda \mathbb{C} = (G \times \mathbb{C})/\sim$$

with $(g, z) \sim (gb^{-1}, \Lambda(b)z)$ for $b \in G_\eta$. As it is well known a section σ of L determines a Λ-equivariant function $f : G \to \mathbb{C}$, i.e., the function f satisfies

$$f(gb) = \Lambda(b^{-1})f(g) \quad \text{for all } b \in G_\eta. \tag{7.7}$$

We denote by $L^\times$ the principal bundle associated to L, i.e., $L^\times = L - \{\text{zero section}\}$. $L^\times$ is a principal $\mathbb{C}^\times$ bundle on $\mathcal{O}$, where $\mathbb{C}^\times$ is the multiplicative group of non-zero complex numbers. The section σ of L defines a function $\sigma^\sharp : L^\times \to \mathbb{C}$ by the relation $\sigma(\pi(y)) = \sigma^\sharp(y) \cdot y$ for $y \in L^\times$. One has

$$f(g) = \sigma^\sharp([g, z])z. \tag{7.8}$$

On $L^\times$, there exists a natural connection α whose construction is detailed in Ref. [18, p. 198]. We recall here the basic properties of α. The group $H = G \times \mathbb{C}^\times$ acts transitively on $L^\times$ by means of the obvious action. The subgroup of isotropy of $u = [e, 1] \in L^\times$ is $H_u = \{(b, \Lambda(b^{-1}))|b \in G_\eta\}$, and its Lie algebra

$$\mathfrak{h}_u = \{(B, -2\pi \mathrm{i}\eta(B))|B \in \mathfrak{g}_\eta\} \subset \mathfrak{g} \oplus \mathbb{C}.$$

Hence $T_u(L^\times) \simeq (\mathfrak{g} \oplus \mathbb{C})/\mathfrak{h}_u$. If $d \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is defined by $d(z) = (1/2\pi \mathrm{i})z$, then (η, d) vanishes on $\mathfrak{h}_u$ and determines an element $\alpha_u \in T_u^*(L^\times)$. The connection α is the H-invariant 1-form on $L^\times$ determined by α_u.

Given X is a vector field on $\mathcal{O}$, by $X^\sharp$ we denote the horizontal lift of X to $L^\times$. From the definition of covariant derivative, one has (see [14, p. 115])

$$(D_X\sigma)^\sharp = X^\sharp(\sigma^\sharp). \tag{7.9}$$

If $\eta' = a\eta$, by (7.3), the character Λ determines a character Λ' of $G_{\eta'}$ putting $\Lambda'(aba^{-1}) = \Lambda(b)$, and the derivative of Λ' is the corresponding $\rho_{\eta'}$. One can define the respective prequantum bundle $L' \equiv L_{\Lambda'}$ on $\mathcal{O} \simeq G/G_{\eta'}$. The identity map of $\mathcal{O}$ gives rise to

$$\phi : gG_{\eta'} \in G/G_{\eta'} \to gaG_\eta \in G/G_\eta.$$

On the other hand, the map $\Phi : [g, z] \in L' \to [ga, z] \in L$ is a well-defined bundle map that covers ϕ. The isomorphism Φ allows us to define a direct correspondence between sections of L and sections of L': for σ section of L, let σ' be the section of L' determined by

$$\Phi \circ \sigma' = \sigma \circ \phi. \tag{7.10}$$

Denoting by f and f' the respective equivariant functions, it is straightforward to prove that $f'(g) = f(ga)$, i.e.,

$$f' = f \circ \mathcal{R}_a \tag{7.11}$$

with $\mathcal{R}_a$ the right multiplication by a in G.

To quantize $\mathcal{O}$ we can start with the prequantum bundle $L = L_\Lambda$, the choice of the maximal torus $T \subset G_\eta$ permits us to define a complex structure I in $\mathcal{O}$ as we explained. The quantization $\mathcal{Q}_I \equiv \mathcal{Q}_I(L)$ of $\mathcal{O}$ obtained using this polarization is the space of holomorphic sections of the line bundle L_I that we will define next. First of all, the character Λ can be extended trivially to the parabolic subgroup P, since P is a semidirect product of $T_{\mathbb{C}}$ and a nilpotent subgroup of $G_{\mathbb{C}}$ (see (7.5)). Then

$$L_I = G_{\mathbb{C}} \times_\Lambda \mathbb{C},$$

in other words, L_I is the line bundle associated to the principal P-bundle $G_{\mathbb{C}} \to G_{\mathbb{C}}/P$ through Λ. The holomorphic sections of L_I can be identified with the holomorphic Λ-equivariant functions on $G_{\mathbb{C}}$.

From the prequantum bundle $L' = L_{\Lambda'}$, by means of the complex structure I on $\mathcal{O}$, one can construct the corresponding quantization $\mathcal{Q}_I(L')$. The map ϕ considered as a map from $\mathcal{O}$ to $\mathcal{O}$ is the identity, hence it is trivially holomorphic w.r.t. I. Therefore, for $\tau \in H^0(L_I)$, the corresponding τ' defined according to (7.10) is also holomorphic. In this way, one has a direct identification between $\mathcal{Q}_I(L)$ and $\mathcal{Q}_I(L')$ which in terms of equivariant functions is given by (7.11).

Now we compare the quantizations $\mathcal{Q}_I(L)$ and $\mathcal{Q}_{\tilde{I}}(L')$, where $\tilde{I}$ is the complex structure on $\mathcal{O}$ defined by the maximal torus $\tilde{T} = aTa^{-1}$. By Proposition 11, the map $l_a : (\mathcal{O}, I) \to (\mathcal{O}, \tilde{I})$ is holomorphic, and l_a induces the mapping

$$\psi : gG_{\eta'} \in G/G_{\eta'} \to a^{-1}ga \in G/G_\eta.$$

On the other hand, the bundle mapping $\Psi : [g, z] \in L' \to [a^{-1}ga, z] \in L$ covers ψ. The section of L' "pull-back" of $\tau \in H^0(L_I)$ by ψ is denoted by $\tilde{\tau}$, i.e., $\Psi \circ \tilde{\tau} = \tau \circ \psi$. Since τ is I-holomorphic, $\tilde{\tau}$ is $\tilde{I}$-holomorphic because of the holomorphy of ψ as mentioned above. Hence we have got a direct identification $\mathcal{Q}_I(L) \simeq \mathcal{Q}_{\tilde{I}}(L')$, which in terms of equivariant functions can be expressed as

$$f \in \mathcal{Q}_I(L) \to \tilde{f} \equiv f \circ \mathcal{L}_{a^{-1}} \circ \mathcal{R}_a \in \mathcal{Q}_{\tilde{I}}(L'), \tag{7.12}$$

where $\mathcal{L}_{a^{-1}}$ is the left multiplication by a^{-1}.

The parallel transport. Let us consider the diffeomorphism $\psi_t : \mu \in \mathcal{O} \to \mathrm{e}^{tA}\mu \in \mathcal{O}$ with $A \in \mathfrak{g}$. This family $\{\psi_t\}$ determines the vector field X_A. By Proposition 11 the complex structure $I_t := \psi_t \cdot I$ is the one defined by the torus $\mathrm{e}^{tA}\, T\, \mathrm{e}^{-tA}$. By (7.2), $\zeta = 2\pi\mathrm{i}h_A\tau - D_{X_A}\tau$ is the natural solution to the transport of τ along I_t. That is, the transported of τ satisfies

$$\tau_t = \tau + t(-D_{X_A}\tau + 2\pi\mathrm{i}h_A\tau) + \mathrm{O}(t^2) \in \mathcal{Q}_{I_t}(L). \tag{7.13}$$

If f is the equivariant function associated to τ, we shall determine the equivariant function associated to τ_t. By (7.8) and (7.9), it will be necessary to determine the horizontal lift $X_A^\sharp$ of X_A. The vector $X_A(g\eta)$ is defined by the curve $\{\mathrm{e}^{tA}\, g\eta\}$ in $\mathcal{O}$. A lift of this curve at the point $[gb, z] \in \pi^{-1}(g\eta)$, where $b \in G_\eta$, is the curve $r(t) := [\mathrm{e}^{tA}\, gb, z_t]$ with $z_t = z\,\mathrm{e}^{tx}$ and $x \in \mathbb{C}$.

$$\dot{r}(0) = [R_A(gb), x] \in T_{[gb,z]}L^\times,$$

where $R_A(gb) \in T_{gb}(G)$ is the value at gb of the right invariant vector field in G determined by $A \in \mathfrak{g}$. By the definition of α and taking into account that $b \in G_\eta$,

$$\alpha(\dot{r}(0)) = \eta(\mathrm{Ad}_{g^{-1}}(A)) + \frac{1}{2\pi\mathrm{i}}x.$$

So $\dot{r}(0)$ is the horizontal lift of $X_A(g\eta)$ if

$$x = -2\pi\mathrm{i}\eta(\mathrm{Ad}_{g^{-1}}A), \tag{7.14}$$

i.e.,

$$X_A^\sharp(gb) = [R_A(gb) - 2\pi\mathrm{i}\eta(\mathrm{Ad}_{g^{-1}}A)] \in T_{gb}L^\times.$$

The action of $X_A^\sharp$ on $\tau^\sharp$ is

$$X_A^\sharp(gb)(\tau^\sharp) = \frac{\mathrm{d}}{\mathrm{d}t}(\tau^\sharp[\mathrm{e}^{tA}gb, z\mathrm{e}^{tx}])|_{t=0}$$

with x given by (7.14). Using (7.8),

$$X_A^\sharp(gb)(\tau^\sharp) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(\mathrm{e}^{tA}\, gb)}{z\,\mathrm{e}^{tx}}\right)\Bigg|_{t=0} = \frac{R_A(gb)(f)\cdot z - f(gb)zx}{z^2}.$$

Then, by (7.9), the Λ-equivariant function associated to $D_{X_A}\tau$ is

$$h(gb) = R_A(gb)(f) + 2\pi\mathrm{i}\eta(\mathrm{Ad}_{g^{-1}}A)f(gb). \tag{7.15}$$

On the other hand, the section $2\pi \mathrm{i} h_A \tau$ corresponds to the Λ-equivariant function $2\pi \mathrm{i} \lambda_A f$ with $\lambda_A(g) = h_A(g\eta) = \eta(\mathrm{Ad}_{g^{-1}} A)$. Consequently, to the right-hand side of (7.13) corresponds the function

$$f - tR_A(f) + \mathrm{O}(t^2). \tag{7.16}$$

As $R_A(g)$ is defined by the curve $\{\mathrm{e}^{tA} g \equiv \mathcal{L}_{\mathrm{e}^{tA}}(g)\}_t$, (7.16) is the expansion of $f \circ \mathcal{L}_{\mathrm{e}^{-tA}}$.

In summary, (7.12) gives the direct identification $f \in \mathcal{Q}_I(L) \to \tilde{f}_t = f \circ \mathcal{L}_{a_t^{-1}} \circ \mathcal{R}_{a_t} \in \mathcal{Q}_{I_t}(L_{\Lambda_t})$, where $a_t = \mathrm{e}^{tA}$ and $\Lambda_t(a_t b a_t^{-1}) = \Lambda(b)$. Using (7.11) one gets the direct identification $h \in \mathcal{Q}_{I_t}(L) \to h \circ \mathcal{R}_{a_t} \in \mathcal{Q}_{I_t}(L_{\Lambda_t})$. By means of the natural transport we have obtained the identification $f \in \mathcal{Q}_I(L) \to f \circ \mathcal{L}_{a_t^{-1}} + \mathrm{O}(t^2) \in \mathcal{Q}_{I_t}(L)$. Hence, the diagram

$$\begin{array}{ccc} \mathcal{Q}_I(L) & \rightarrow & \mathcal{Q}_{I_t}(L) \\ \searrow & & \swarrow \\ & \mathcal{Q}_{I_t}(L_{\Lambda_t}) & \end{array}$$

is commutative up to order t^2.

Proposition 12. *The identification $\mathcal{Q}_I(L) \simeq \mathcal{Q}_{I_t}(L)$ given by the natural transport is compatible with the direct identifications $\mathcal{Q}_I(L) \simeq \mathcal{Q}_{I_t}(L_{\Lambda_t})$ and $\mathcal{Q}_{I_t}(L) \simeq \mathcal{Q}_{I_t}(L_{\Lambda_t})$.*

8. Relation with Berry phase

For an antiholomorphic submanifold P (relative to I) that undergoes the cyclic evolution $P_t = \psi_t(P)$, the corresponding Berry phase is defined [27]. Here we study the relation between this phase and the holonomy of the natural transport $\tau_{|P} \to \tau_{t|P_t}$ along the curve $I_t = \psi_t \cdot I$.

The connection on the $\mathbb{C}^\times$-principal bundle $L^\times = L - \{\text{zero section}\}$, associated to the prequantum bundle L, will be denoted by α. Given $c \in \mathbb{C}$, the vertical vector field on $L^\times$ generated by c will be denoted by W_c. That is, $W_c(q)$ is the vector defined by the curve in $L^\times$ given by $q\mathrm{e}^{2\pi \mathrm{i} ct}$.

Remark. Let V be a $\mathbb{C}$-vector space; if we consider V as a manifold for $c \in \mathbb{C}$ and $v \in V$, the curve $\{\mathrm{e}^{2\pi \mathrm{i} ct} v\}$ defines a vector $W_c(v) \in T_v V$. Then from the natural isomorphism $T_v V \simeq V$, one obtains the following evident equality in $T_v V$:

$$W_c(v) = 2\pi \mathrm{i} c \cdot v. \tag{8.1}$$

If P is an antiholomorphic integral submanifold relative to I and $\tau \in \mathcal{Q}_I$, the condition $D''\tau = 0$ implies $D_v \tau = 0$ for all $v \in TP$. So $\tau_{|P}$ is a parallel section of $L_{|P}$; consequently if $\tau \neq 0$, then $\tau(p) \neq 0$ for all $p \in P$ and $\tau(P) \subset L^\times$ is a Planckian submanifold [25] of $L^\times$ over P.

Lemma 13. *If $X \in T_m P$ and $\tau \in \mathcal{Q}_I$, the vector $\tau_*(X) \in T_q L^\times$, where $q = \tau(m)$, satisfies $\tau_*(X) = H(X)(q) + (D_X \tau)(m)$ with H(X)(q) the horizontal lift of X at the point q.*

Proof. We set $Z := \tau_*(X) \in T_q(L^\times)$. Let Z_1 and Z_2 be the horizontal and vertical parts of $\tau_*(X)$, respectively. We want to prove that $Z_1 = H(X)(q)$ and $Z_2 = (D_X\tau)(m)$. Denoting by π the projection from the bundle to M,

$$\pi_*(Z) = X = \pi_*(Z_1) + \pi_*(Z_2) = \pi_*(Z_1).$$

Since Z_1 is horizontal and $\pi_*(Z_1) = X$, it turns out that Z_1 is the horizontal lift of X at the point q. On the other hand, $Z_2 = W_c(q)$ for some $c \in \mathbb{C}$.

$$(D_X\tau)(m) = (\tau^*\alpha)(X)\cdot\tau(m) = \alpha(\tau_*(X))\cdot q,$$

and $\alpha(\tau_* X) = \alpha(Z_2) = \alpha(W_c(q)) = 2\pi\mathrm{i}c$. Hence by the above remark $(D_X\tau)(m) = W_c(q) = Z_2$. □

Let $\{\psi_t\}$ be a family of Hamiltonian symplectomorphisms of (M,ω) with $\psi_0 = \mathrm{id}$ generated by the time-dependent Hamiltonian f_t. By Proposition 8 the transport along $I_t := \psi_t\cdot I$ has the natural solution $\zeta = -2\pi\mathrm{i}f_t\tau_t - D_{X_t}\tau_t$. If P is an antiholomorphic integral submanifold of I, then $\psi_t(P) =: P_t$ is also an antiholomorphic integral submanifold of I_t. Assuming that $\tau \in \mathcal{Q}_I$ is different from 0, $\tau_t(P_t)$ is a Planckian submanifold of $L^\times$ on P_t, τ_t being the transport of τ along I_t by means of the natural transport.

Given $p \in P$, one can consider in $L^\times$ the following curve:

$$t \to \tau_t(\psi_t(p)).$$

Proposition 14. *The tangent vector defined by* $\{\tau_t(\psi_t(p))\}_t$ *at* $q = \tau_u(\psi_u(p))$ *is*

$$H(X_u)(q) + W_{-f_u(\pi(q))}(q).$$

Proof. For t in a small neighbourhood of u, as

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = -2\pi\mathrm{i}f_t\tau_t - D_{X_t}\tau_t,$$

then

$$\begin{aligned}\tau_t(\psi_t(p)) &= \tau_u(\psi_t(p)) - (t-u)(2\pi\mathrm{i}f_u(\psi_t(p))\tau_u(\psi_t(p)) \\ &\quad +(D_{X_u}\tau_u)(\psi_t(p))) + \mathrm{O}((t-u)^2).\end{aligned}$$

This curve defines at $t = u$ the following vector of $T_qL^\times$:

$$(\tau_u)_*(X_u(s)) - (2\pi\mathrm{i}f_u(s)\tau_u(s) + (D_{X_u}\tau_u)(s)), \tag{8.2}$$

where $s := \psi_u(p)$. As $\tau_u(s) = q$ by Lemma 5, $(\tau_u)_*(X_u(s)) = H(X_u(s))(q) + (D_{X_u}\tau_u)(s)$. By (8.1), expression (8.2) is equal to

$$H(X_u(\pi(q)))(q) - W_{f_u(\pi(q))}(q).$$

In short, the tangent vector at q defined by the curve considered is $H(X_u) + W_{-f_u}$. □

Now let $\{\psi_t\}_{t\in[0,1]}$ be a closed smooth path in the group $\mathrm{Ham}(M)$ with $\psi_0 = \psi_1 = \mathrm{id}$ generated by the differentiable family of Hamiltonians $\{f_t\}$. Moreover, we assume that the vector field X_t is of type $(1,0)$ on P_t. Thus, we have a curve $\{P_t\}$ which is an isodrastic deformation of P; an isodrastic loop of submanifolds $\{P_t\}$ in fact. Given $\tau \in \mathcal{Q}_I$, we call the transport $\tau_{|P} \to \tau_{t|P_t}$ the transport of τ along the isodrast P_t.

We recall some results of Weinstein about the Berry phase (for details see [27, p. 142]). Let ϵ_t be a smooth density on P_t such that $\int_{P_t} f_t \epsilon_t = 0$. Let $\{r_t\}$ be the family of isomorphisms of $(L^{\times}_{|P}, \alpha)$ to $(L^{\times}_{|P_t}, \alpha)$ determined by $\{f_t\}$, i.e., the isomorphisms generated by the vector fields

$$H(X_t) + W_{-f_t}. \tag{8.3}$$

The submanifold $r_1(\tau(P))$ "differs" from $\tau(P)$ by an element $\theta \in \mathbb{C}^{\times}$, i.e.,

$$r_1(\tau(P)) = \theta\tau(P). \tag{8.4}$$

If we denote by hol the holonomy on P defined by the connection α, $\mathrm{hol} : \pi_1(P) \to \mathbb{C}^{\times}$, and by $\gamma : z \in \mathbb{C}^{\times} \to z|z|^{-1} \in U(1)$. Then the Berry phase of the family (P_t, ϵ_t) of weighted submanifolds is the class of $\gamma(\theta)$ in the quotient $U(1)/(\mathrm{Im}(\gamma \circ \mathrm{hol}))$. Up to here the results are of Weinstein.

Theorem 15. *The Berry phase of (P_t, ϵ_t) is the class in $U(1)/(\mathrm{Im}(\varphi\circ\mathrm{hol}))$ of the holonomy of the natural transport along the isodrastic loop P_t.*

Proof. Given $p \in P$, by Proposition 14 and (8.3) the curves in $L^{\times}\{\tau_t(\psi_t(p)\}$ and $\{r_t(\tau(p))\}_t$ define the same vector field. As they take the same value for $t = 0$, it turns out that $r_t(\tau(p)) = \tau_t(\psi_t(p))$ for all $p \in P$; hence the above complex θ in (8.4) is determined by

$$\tau_1(p) = \theta\tau(p). \tag{8.5}$$

The natural transport along the loop I_t carries $\tau \in \mathcal{Q}_I$ to τ_1. Since $\tau_{1|P}$ and $\tau_{|P}$ are parallel w.r.t. the connection α on $L^{\times}$, one has

$$\tau_{1|P} = \kappa\tau_{|P} \tag{8.6}$$

with $\kappa \in \mathbb{C}^{\times}$, assuming P is connected. (Note that κ is independent of $\tau \in \mathcal{Q}_I$.) Hence κ can be regarded as the holonomy of the natural transport along the isodrastic loop of submanifolds P_t. From (8.5), we conclude that $\theta = \kappa$. $\square$

Appendix A. Complex coordinates on M_I

Given $I \in \mathcal{K}$, we have the direct sum decomposition $\mathcal{A} = \oplus\mathcal{A}^{p,q}$ of the space of C^{∞} differential forms on M; and for fixed p we have the operators

$$\bar{\partial}_j : \mathcal{A}^{p,j} \to \mathcal{A}^{p,j+1},$$

defined by $\bar{\partial}_j = \pi_{p,j+1}d$, where $\pi_{p,j+1}$ is the corresponding projector.

On the other hand, the metric $g_I = \omega(., I.)$ determines the respective Hodge $\star$-operator. We have also defined the inner product of p-forms:

$$(\varphi', \varphi) := \int \varphi' \wedge \star\bar{\varphi}.$$

By δ_j we denote the adjoint of $\bar{\partial}_j$, and Δ_j is the corresponding Laplacian operator. The integrability condition for I implies $\bar{\partial}_j \bar{\partial}_{j-1} = 0$, so we have the respective elliptic complex $\{\mathcal{A}^{p,*}, \bar{\partial}\}$, and the operators δ, Δ and the Green's operator G defined by natural grading.

We now summarize the construction of a local holomorphic coordinate system for M_I achieved in Ref. [17]. Let $x_1, \ldots, x_{2n}$ be real coordinates on a neighbourhood W of $x \in M$ with origin at x, such that $x_i(W) \supset [-1, 1]$ and

$$\left(\frac{\partial}{\partial x_{2i}}\right)_x = I_x \left(\frac{\partial}{\partial x_{2i-1}}\right)_x .$$

One defines the functions h_i^k on W by

$$I\left(\frac{\partial}{\partial x_i}\right) = \sum_k h_i^k \left(\frac{\partial}{\partial x_i}\right). \tag{A.1}$$

Let B be the unit ball in $\mathbb{R}^{2n}$. By B_t we denote the almost complex manifold whose underlying differentiable manifold is B, and whose almost complex structure $\hat{J}_t$ is given by

$$\hat{J}_t\left(\frac{\partial}{\partial x_i}\right) := \sum_k h_i^k(tx_1, \ldots, tx_{2n}) \left(\frac{\partial}{\partial x_k}\right) \tag{A.2}$$

for $t \in [0, 1]$ (here the x_i's are the usual coordinates on $\mathbb{R}^{2n}$). We denote by δ_t, G_t and $\bar{\partial}_t$ the obvious operators relative to the structure $\hat{J}_t$.

The functions

$$z_t^i := z^i - \delta_t G_t \bar{\partial}_t z^i, \tag{A.3}$$

with $z^j = x_{2j-1} + \mathrm{i}x_{2j}$, are holomorphic on B_t; there is a neighbourhood V of the origin and $T > 0$ such that $\{z_T^j\}_j$ is a coordinate system on V. Let $U' := T \cdot V$, then for $(x_1, \ldots, x_{2n}) \in U'$ one defines

$$u^i(x_1, \ldots, x_{2n}) := z_T^i \left(\frac{x_1}{T}, \ldots, \frac{x_{2n}}{T}\right). \tag{A.4}$$

The set $\{u^j\}_j$ is a holomorphic system for M_I about x [17, p. 146]. So far the construction of Kohn is over.

Let $\mathcal{T}$ be a topological space and $\lambda \in \mathcal{T} \to J^\lambda \in \mathcal{K}$ a continuous map, where $\mathcal{K}$ is endowed with the differential structure mentioned in Section 2. Therefore given $\mu \in \mathcal{T}$, $l \geq 0$ and $\epsilon > 0$, there is a neighbourhood of μ in $\mathcal{T}$ such that $\|J^\mu - J^\lambda\|_l < \epsilon$ for all λ in this neighbourhood. (Here $\| \, \|_l$ denotes the corresponding Sobolev norm).

We denote

$$T_{(\lambda)}^{1,0}(M) := \{X - \mathrm{i}J^\lambda X : X \in T(M)\}.$$

Since $\{J^\lambda\}_\lambda$ is continuous, one can construct local orthonormal frames $\{e_j^\lambda\}_j$ and $\{\bar{e}_j^\lambda\}$ for $T_{(\lambda)}^{1,0}(M)$ and $T_{(\lambda)}^{0,1}(M)$, respectively, such that for each j the family $\{e_j^\lambda\}_\lambda$ is continuous w.r.t. $\| \, \|_l$. Therefore, there are local frames $\{w_i^\lambda\}_i$ for $\mathcal{A}_{(\lambda)}^{p,q}$ which depend continuously on λ. Denoting by $\bar{\partial}^\lambda := \pi^\lambda d$, where π^λ is the respective projection, then the family $\{\bar{\partial}^\lambda w_i^\lambda\}_\lambda$ is also continuous. So we have the following proposition.

Proposition A.1. *Given a continuous family $\{J^\lambda\}_\lambda$ in $\mathcal{K}$, there exists a local frame $\{w_i^\lambda\}$ of $\mathcal{A}_{(\lambda)}^{p,q}$ such that the family $\{\bar{\partial}^\lambda w_i^\lambda\}_\lambda$ is continuous w.r.t. the Sobolev norms $\| \, \|_l, l \geq 0$.*

If in the statement of Proposition A.1 we exchange the operator $\bar{\partial}^\lambda$ with δ^λ or Δ^λ or G^λ, the new propositions remain true.

Fixed $\mu \in \mathcal{T}$ and setting $J := J^\mu$, let $\{v_1, \ldots, v_{2n}\}$ be a local basis of TM defined on a neighbourhood $\tilde{W}$ of x, such that $v_{2j}(x) = J_x(v_{2j-1}(x))$, $j = 1, \ldots, n$, and let $\{x_1, \ldots, x_{2n}\}$ be real coordinates which satisfies: (a) $x_j(\tilde{W}) \supset [-1, 1]$, (b) $x_j(x) = 0$ and (c) $\partial/\partial x_j = v_j$. For λ in a neighbourhood $\mathcal{V}$ of μ we can choose a local basis $\{v_j^\lambda\}_j$ on $\tilde{W}$ with $v_{2j}^\lambda(x) = J_x(v_{2j-1}^\lambda(x))$, and coordinates $\{x_j^\lambda\}_j$ which satisfy the conditions (a) and (b), and the property (c) w.r.t. $\{v_j^\lambda\}_j$. The coordinates x_j^λ can be chosen so that for each j the family $\{x_j^\lambda\}_\lambda$ is continuous w.r.t. the norm of the Sobolev spaces $H_l(\tilde{W})$. By (A.1) the corresponding family $\{h_j^{\lambda k}\}_\lambda$ is also continuous. For fixed λ, the functions $\{h_j^{\lambda k}\}_\lambda$ define a complex structure $\hat{J}_t^\lambda$ by means of (A.2). Moreover, the family $\{\hat{J}_t^\lambda\}_\lambda$ is continuous w.r.t. the norm of Sobolev spaces $H_l(\mathrm{End}\, TB), l \geq 0$. Now we can construct a holomorphic coordinate system for M_{J^λ} using the $\{\hat{J}_t^\lambda\}_\lambda$, and by Proposition A.1, (A.3) and (A.4) we can state the following proposition.

Proposition A.2. *Given $x \in M$ and $\mu \in \mathcal{T}$, there exist a neighbourhood U of x in M, a neighbourhood $\mathcal{V}$ of μ in $\mathcal{T}$, and for each $\lambda \in \mathcal{V}$ a local holomorphic coordinate system $\{u_j^\lambda\}_j$ for M_{J^λ} defined on U, such that for every j, $\{u_j^\lambda\}_\lambda$ is a continuous family w.r.t. the norm in $H_l(U)$, $l \geq 0$.*

References

[1] S. Axelrod, S. Della Pietra, E. Witten, Geometric quantization of Chern–Simons gauge theory, J. Diff. Geom. 33 (1991) 787–902.
[2] E. Ebin, The manifold of Riemannian metrics, Proc. Symp. Pure Math. AMS 15 (1970) 11–40.
[3] G. Faltings, Stable G-bundles and projective connections, J. Alg. Geom. 2 (1993) 507–568.
[4] A. Fujiki, G. Schumacher, The moduli space of Kähler structures on a real compact symplectic manifold, Publ. RIMS Kyoto Univ. 24 (1988) 141–168.
[5] W. Fulton, J. Harris, Representation Theory, Springer, New York, 1991.
[6] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.
[7] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press, Cambridge, 1984.
[8] R.G. Gunning, Riemann Surfaces and Generalized Theta Functions, Springer, Berlin, 1976.
[9] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer, Berlin, 1978.
[10] N. Hitchin, Flat connections and geometric quantization, Commun. Math. Phys. 131 (1990) 347–380.

[11] D. Husemoller, Fibre Bundles, Springer, New York, 1975.
[12] A.A. Kirilov, Elements of the Theory of Representations, Springer, Berlin, 1976.
[13] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Iwanami Shoten Publishers/Princeton University Press, Princeton, NJ, 1987.
[14] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry I, Wiley, New York, 1963.
[15] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry II, Wiley, New York, 1969.
[16] K. Kodaira, Complex Manifolds and Deformation of Complex Structures, Springer, New York, 1986.
[17] J.J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds I, Ann. Math. 78 (1963) 112–148.
[18] B. Kostant, Quantization and unitary representations, in: C.T. Taam (Ed.), Lectures in Modern Analysis III, Lecture Notes in Mathematics, Vol. 170, Springer, Berlin, 1970, pp. 87–208.
[19] H. Lange, C. Birkenhake, Complex Abelian Varieties, Springer, Berlin, 1992.
[20] R. Narasimhan, Analysis on Real and Complex Manifolds, North-Holland, Amsterdam, 1985.
[21] H. Omori, On the group of diffeomorphisms of a compact manifold, Proc. Sympos. Pure Math. 15 (1968) 167–183.
[22] H. Omori, Infinite-dimensional Lie transformation groups, Lecture Notes in Mathematics, Vol. 427, Springer, Berlin, 1974.
[23] T.R. Ramadas, Faltings' construction of the K–Z connection, Commun. Math. Phys. 196 (1998) 133–143.
[24] B. Simon, Representations of Finite and Compact Groups, American Mathematical Society, Providence, RI, 1996.
[25] J.-M. Souriau, Structure des Systèmes Dynamiques, Dunod, Paris, 1970.
[26] A. Viña, Bundle of quantizations of a symplectic torus, Lett. Math. Phys. 36 (1996) 231–245.
[27] A. Weinstein, Connections of Berry and Hannay type for moving Lagrangian submanifolds, Adv. Math. 82 (1990) 133–159.
[28] R.O. Wells, Differential Analysis on Complex Manifolds, Springer, New York, 1980.
[29] N.M.J. Woodhouse, Geometric Quantization, Clarenton Press, Oxford, 1992.